# Transitional Dynamics in the Uzawa-Lucas Model of Endogenous Growth

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**Abstract**

We introduce an easy way of analyzing the transitional dynamics of the Uzawa-Lucas endogenous growth model. We use the value function approach to solve both the social planner's optimization problem and the representative agent's optimization problem in the decentralized economy. The complexity of the Hamilton-Jacobi-Bellman equation is significantly reduced to a one-dimensional initial value problem for an ordinary differential equation. This approach allows us to find the optimal controls for the non-concave Hamiltonian in the centralized economy and to detect multiple transition paths in the decentralized economy for a large external effect, which are hidden when using the maximum principle.

We simulate the global transitional dynamics towards the balanced growth path. The adjustment of the model's state variable turns out to accelerate along the transition paths. By the asymmetry of the sectors an until now unknown feature is predicted for the adjustment in the output growth rate. Its relative speed follows a hump-shaped course: Starting from a relative scarcity in physical capital, the growth rate of output decelerates first before it starts rising again.

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# 1 Introduction

Endogenous growth theory studies macroeconomic models where the growth rate of output and other key statistics are determined within the model. A very intuitive and widely known endogenous growth model is the AK model. AK denotes the aggregate production function where the positive constant $A$ stands for the level of technology and the variable $K$ captures capital in a broad sense, i.e. it includes for example human capital and infrastructure. Non-diminishing returns then induce growth[1].

The AK model presented here is based on the Uzawa (1965) and Lucas (1988) endogenous growth model. Its main feature is the fact that the representative agent – or the social planner – has to allocate his or her human capital between two production sectors. First, there is a goods sector where a single good usable for consumption and physical capital investment is produced. This sector exhibits a production technology that uses human and physical capital. Second, there is a schooling sector where the representative agent produces his stock of human capital. Here, human capital is the only production factor. In short: the agent has to 'learn or to do' (Chamley (1993)). The two sector structure considered here constitutes the main difference to the usual AK model where intellectual and physical capital are summed-up in the single variable K.

In his seminal paper Lucas (1988) argues that the agents' average level of human capital contributes to the productivity of all factors of goods production. In a decentralized economy the individual's accumulation of human capital has no appreciable influence on the economy's average level of human capital. In a decentralized symmetric equilibrium no one therefore takes this effect into account when deciding how to allocate her human capital. The mechanism behind this market failure is analogous to that in a Nash game producing the prisoner's dilemma. When deciding how much to invest in human capital the agents have no incentive to take their influence on the average level of human capital into account. This results in a non-efficient equilibrium in the sense that the agents' discounted utility could be higher without making a single agent worse off. As a result the solution for the centralized economy where this external effect is exploited by the planner differs from the decentralized case.

The failure of the second welfare theorem in the Uzawa-Lucas endogenous growth model has inspired Benhabib and Perli (1994) to study parameter spaces that give rise to multiple equilibria. Their solution method is however not applicable in the centralized case since the Hamiltonian is not concave in the presence of a positive external effect. In general the centralized case has been neglected in the literature. The aim of our paper is to analyze the centralized case as well as the decentralized case. The theoretical model considered here differs from Benhabib and Perli (1994) only in the choice of the utility function. We assume logarithmic preferences which imply that the constant intertemporal elasticity of substitution is equal to one. This assumption reduces the number of parameters by one and simplifies calculations. Nevertheless the balanced growth

[1]The interested reader is referred to Barro and Sala-i-Martin (1995) for a general treatment of growth theory and to Aghion and Howitt (1998) for endogenous growth theory.

path implications are analogous to those in the more general case. Mulligan and Sala-i-Martin (1993) stress that a parametrization where the inverse of the intertemporal elasticity of substitution is bigger than the output elasticity of physical capital in the goods sector is empirically the most relevant case. Xie (1994) studies the special constellation where both parameters are equal. Xie also focusses on the decentralized case and the chosen parameters allow him to derive the explicit dynamics. Hartley and Rogers (2003) solve an Arrow and Kurz (1970) type of a two sector growth model in closed form after introducing a stochastic disturbance in the law of capital accumulation.

The model's parametrization chosen here makes it possible to write down an explicit functional form of a solution to the resulting Hamilton-Jacobi-Bellman equation. This allows us to follow the value function approach rather than the Pontryagin maximum principle adopted by the above-mentioned papers. The knowledge of an explicit functional form solving the Hamilton-Jacobi-Bellman equation facilitates our analysis. We are able to reduce the dimension of the optimal decision rules and thereby simplify our analysis. Our "candidate" function, however, is not the planner's value function except for one specific initial value. We show that at this particular value a saddle point behavior occurs. An application of the candidate function outside the steady state yields an unstable solution branch. For the stable solution through this saddle point, which is the true value function, an analytical expression is unknown, but by suitable transformation the numerical analysis becomes easily tractable. For this we take advantage of the fact that we already know the explicit form of the value function at one point. Finally, it turns out that the transitional paths of the optimal decision rules are determined by an ordinary differential equation in the model's parameters. The global character of our method allows to study the model far away from the balanced growth path. For the decentralized economy we are able to find multiple transition paths so far undetected in the literature.

Our simulations of the model's solution show that the model implies hump-shaped adjustment of output growth rates. When physical capital is relatively scarce the growth rate of output is very high but declining. The growth rate even falls below the balanced growth path value before it starts rising again and finally converges to the balanced growth rate. We argue that the model's inherent asymmetry is responsible for this feature. By shifting human capital from the educational sector to the goods sector it is possible to transform human capital into physical capital. Since the schooling technology is linear in human capital a transformation in the opposite direction is not possible.

The paper is organized as follows. Section 2 introduces the model. In Section 3 we present our strategy of solving the central planner's problem. In Section 4 we slightly adjust our strategy from the previous section in order to cope with the decentralized economy. The basic ideas remain similar, but a higher analytical and numerical effort is needed because a fixed point problem occurs. In Section 5 we present simulation results implying the hump-shaped course of output growth. Section 6 concludes. Appendix A contains proofs of statements omitted in the paper. Appendix B discusses further simulation results.

# 2 The model

This section introduces the theoretical model. We assume a closed economy populated by a large number of identical infinitely-lived agents. Firms are producing a single good and there is a schooling sector providing educational services.

## 2.1 The households

We assume that the population is constant and normalized to one. The representative agent has logarithmic preferences over consumption streams

$$U = \int_{t=0}^{\infty} e^{-\rho t} \log\left(c_t\right) dt, \tag{1}$$

where $c_t$ is the level of consumption at time $t$ and $\rho > 0$ is the subjective discount rate. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. Agents have a fixed endowment of time, which is normalized as a constant flow of one unit. The variable $u_t$ denotes the fraction of time allocated to goods production at time $t$. Furthermore, as agents do not benefit from leisure the whole time budget is allocated to the two production sectors. The fraction $1 - u_t$ of time is spent in the schooling sector. Note that in any solution the condition

$$u_t \in [0, 1] \tag{2}$$

has to be fulfilled. The variables $c_t$ and $u_t$ are the two control variables of the agent. When maximizing the discounted stream of utility the agent has to pay attention to the following budget constraint:

$$r_t k_t + w_t u_t h_t = c_t + \dot{k}_t, \qquad \forall t \geq 0,$$

where $\dot{k}_t$ is the rate of change of the agent's physical capital stock $k_t$. Since we abstract from depreciation, this rate corresponds to the agent's net investment in physical capital. The variables $r_t$ and $w_t$ are market clearing factor prices, i.e. the real interest rate and the real wages, respectively. These prices are taken as given by the representative agent.

The left hand side describes the stream of income derived from physical capital plus the income stream stemming from human capital $h_t$ used in the goods producing sector, i.e. $h_t u_t$. We assume that the initial values of $k_t$ and $h_t$ are strictly positive. On the right hand side the spending of the agent's earnings appears, which he can either consume or invest. Another constraint the agent has to keep in mind is the evolution of his stock of human capital when allocating $1 - u_t$ to the schooling sector.

## 2.2 The schooling sector

The creation of human capital is determined by a linear technology in human capital only:

$$\dot{h}_t = B\left(1 - u_t\right) h_t, \tag{3}$$

where we assume that $B$ is positive[2]. This technology together with constraint 2 implies that human capital will never shrink, i.e. the growth rate $\dot{h}$ must be non-negative. If we set $u_t$ in equation (3) equal to zero, we get the potential growth rate of human capital. If we set $u_t$ equal to one, a stagnation of human capital follows. The schooling technology implies that the potential marginal and average product coincide and are equal to $B$ whereas the realized marginal and average product are equal to $B(1-u_t)$. Note that we abstract from depreciation.

## 2.3 The goods sector

We assume an infinitely large number of profit maximizing firms producing a single good. They are using a Cobb Douglas technology in the two inputs physical and human capital. The level of human capital utilized in goods production equals the total level of the stock of human capital multiplied by the fraction of time spent in the goods sector at time t. Total factor productivity $A$ is enhanced by the external effect $\gamma$ of the economy's average stock $h_{a,t}$ of human capital. Hence, output $y_t$ is determined by:

$$y_t = Ak_t^{\alpha}\left(h_t u_t\right)^{1-\alpha} h_{a,t}^{\gamma}.$$

The parameter $\alpha$ is the output elasticity of physical capital and we assume $\alpha \in (0,1)$. We further assume that the exponent $\gamma$ is nonnegative. If we set $u_t$ equal to one, we get the potential output in the goods sector. Since all agents are homogeneous, the economy's average level of human capital must equal the representative agent's level of human capital at any point in time:

$$h_t = h_{a,t}, \quad \forall t \geq 0. \tag{4}$$

The firm has to rent physical and human capital on perfectly competitive factor markets. In the decentralized economy the representative firm's profit $\Pi$ in period $t$ is given by:

$$\Pi\left(k_t, h_t; h_{a,t}\right) = Ak_t^{\alpha}\left(h_t u_t\right)^{1-\alpha} h_{a,t}^{\gamma} - r_t k_t - w_t u_t h_t,$$

where the semicolon indicates that the economy's average level of human capital is treated as exogenous by the firms (and the agents). The first order conditions for the profit maximizing factor demands are:

$$r_t \equiv \frac{\partial y_t}{\partial k_t} = \frac{\alpha y_t}{k_t} \quad \text{and} \quad w_t \equiv \frac{\partial y_t}{\partial\left(u_t h_t\right)} = \frac{(1-\alpha)\, y_t}{u_t h_t}.$$

The market clearing factor prices ensure that the zero profit condition holds. Inserting the prices into the agent's budget constraint yields the same restriction as would have been imposed by the central planner:

$$y_t = c_t + \dot{k}_t, \qquad \forall t \geq 0,$$

[2]The case when $B$ equals 0 corresponds to the neoclassical growth model.

Note that by consuming more than current production it is possible to disinvest in physical capital, i.e. the growth rate of physical capital turns negative. Having introduced the basic features of our model we now turn to the social planner's problem and its solution.

# 3 The centralized economy

In a centralized economy the planner exploits the equality condition of equation (4). His dynamic optimization problem (DOP) is given by:

$$U = \max_{\{c_t, u_t\}_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-\rho t} \log(c_t)\, dt,$$

with respect to the state dynamics

$$\begin{aligned}
\dot{k}_t &= A k_t^{\alpha} u_t^{1-\alpha} h_t^{1-\alpha+\gamma} - c_t, && \forall\, t \geq 0, \\
\dot{h}_t &= B (1 - u_t)\, h_t, && \forall\, t \geq 0, \\
k_t &\geq 0 \quad \text{and} \quad h_t \geq 0 && \forall\, t \geq 0.
\end{aligned}$$

The initial values $k_0$, $h_0 > 0$ are assumed to be given. Requiring the initial stocks of capital to be strictly positive ensures an interior solution and rules out trivial solutions. Since we assume a Cobb Douglas production technology and logarithmic utility, this restriction will be satisfied automatically under optimal controls.

Although we stated the above DOP for the central planner, it turns out that the representative agent's DOP in the decentralized economy is very similar. The only difference between the two is the use of equation (4). The central planner uses this information before deriving the first order conditions whereas the representative agent uses it thereafter. This difference reflects the inefficient incentive structure described in the introduction.

In Section 3.1 we solve the central planner's DOP presented above. Using homogeneity in the initial conitions, we are able to reduce the corresponding Hamilton-Jacobi-Bellman (HJB) equation to only one implicit ordinary differential equation. We can give an explicit solution of the HJB equation, but this "candidate" is not the planner's value function except for one specific initial value. We show that at this point a saddle point behavior of the HJB equation occurs and that it describes the balanced growth path of the economy. An application of the candidate function to the left of the steady state yields an unstable solution branch giving non-admissible controls, that is physical capital tends to minus infinity for $t \to \infty$. Applying the candidate to the right of the steady state finally results in an excess accumulation of physical capital indicating that the agents' consumption level is dynamically inefficient. For the stable solution through this saddle point, which is the true value function, an analytical expression is unknown.

In Section 3.2, we transform the problem of determining the value function into an initial value problem for an explicit one-dimensional ordinary differential equation. The linear approximation at the saddle point is given in terms of the parameters. Moreover, the explicit form makes it possible to apply the classical Euler scheme in order to determine the solution numerically[3].

### 3.1 The social planner's optimization problem

In the DOP, the two control functions $c_t$ and $u_t$ are chosen by the social planner given the set of admissible controls

$$(c_t, u_t)_{t\geq 0} \in \mathcal{X} := \{(f,g) : [0,\infty) \to X \mid f,\, g \text{ locally bounded and measurable}\}$$

with $X := [0,\infty) \times [0,1]$. Using the logarithmic utility function and the exponential discount rate, the planner defines the representative agent's value function:

$$V(k_0, h_0) := \max_{(c,u)\in\mathcal{X}} \begin{cases} \int_0^\infty \log(c_t) e^{-\rho t}\, dt, & \tau = \infty \\ -\infty, & \tau < \infty, \end{cases}$$

where $\tau$ denotes the stopping time

$$\tau := \inf\{t \geq 0 \mid k(t) = 0\}.$$

This is a classical optimal control problem with infinite horizon (Fleming and Soner, 1995, Section I.7). However, the results derived there are not directly applicable because $x \mapsto x^p$ for $p \in (0,1)$ and $x > 0$ is only locally Lipschitz continuous and we allow $V = -\infty$. Nevertheless, it turns out that the optimal controls imply dynamics where the state variables are bounded away from zero so that $\tau = \infty$ holds and the above-mentioned conditions are satisfied. In order to determine the value function, we write down the HJB equation for the value function $V(\cdot)$ at $k, h > 0$ and $t \geq 0$:

$$\rho V = \max_{(c,u)\in X} \big(\log(c) + V_k(Ak^\alpha u^{1-\alpha} h^{1-\alpha+\gamma} - c) + V_h B(1-u)h\big).$$

Here, $V_k$ and $V_h$ denote the partial derivatives $\frac{\partial V}{\partial k}$ and $\frac{\partial V}{\partial h}$, respectively and can be interpreted as the shadow prices of relaxing the corresponding constraints. Recall that in the case of an infinite time horizon, time-homogeneous equations and an exponential discount rate, the HJB equation simplifies to a differential equation that is independent of time. Observe further that the planner has already inserted the symmetric equilibrium condition stated in equation (4).

We determine the maximum by looking at the first order necessary conditions. The implied optimal controls are given by:

$$c^* = V_k^{-1}, \tag{5}$$

$$u^* = \left(\frac{A(1-\alpha)V_k}{BV_h}\right)^{1/\alpha} \frac{k}{h^{(\alpha-\gamma)/\alpha}}. \tag{6}$$

[3]This scheme is provided by standard mathematical software packages.

The planner chooses the consumption stream such that the marginal utility is equal to the marginal change of wealth with respect to physical capital. The optimal allocation of human capital between the two sectors is determined by the weighted ratio of the marginal changes in goods and human capital production due to a marginal shifting of the human capital allocation. The respective weights are the shadow prices of the corresponding state variable. Then this ratio is raised to the power of the inverse of the output elasticity of physical capital.

Since the value function $V(\cdot)$ is obviously increasing in its arguments, the relation found for $c^*$ ensures that the consumption rate is positive. Equally, $u^* \in (0,\infty)$ holds, but $u^* > 1$ may well occur. For the moment, let us suppose that the values $(u^*, c^*)$ found above are in $X$. Then the HJB equation becomes:

$$\rho V + 1 = -\log(V_k) + \alpha k \left(AV_k h^\gamma\right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{BV_h}\right)^{\frac{1-\alpha}{\alpha}} + BV_h h. \tag{7}$$

In fact the HJB equation is homogeneous in the initial conditions. This allows us to follow Mulligan and Sala-i-Martin (1993) in defining a so-called state-like variable $x_t := k_t h_t^{-(1-\alpha+\gamma)/(1-\alpha)}$. Note that $Ax_t^{1-\alpha}$ is the potential output to capital ratio. The introduction of $x_t$ reduces the complexity of the problem by one dimension. Its dynamics are given by

$$\dot{x}_t = Ax_t^\alpha u_t^{1-\alpha} - c_t x_t k_t^{-1} - \tfrac{1-\alpha+\gamma}{1-\alpha} B(1-u_t)x_t \tag{8}$$

Introducing the control-like variable $q_t := c_t x_t k_t^{-1}$, we see that the evolution of $x_t$ is completely described by $x_t$, $u_t$ and $q_t$. For any initial state $(\tilde{k}_0, \tilde{h}_0)$ with $\tilde{x}_0 := \tilde{k}_0 \tilde{h}_0^{-(1-\alpha+\gamma)/(1-\alpha)} = x_0$ we are led to apply the same controls $\tilde{u}_t = u_t$ and $\tilde{q}_t = q_t$. The only difference is that the consumption rate $\tilde{c}_t$ differs from $c_t$ by the factor $(\tilde{h}_0/h_0)^{(1-\alpha+\gamma)/(1-\alpha)}$. Any solution $V(k,h)$ can thus be deduced from $V(x,1) =: f(x)$ via

$$V(k,h) = f(kh^{-(1-\alpha+\gamma)/(1-\alpha)}) + \tfrac{1-\alpha+\gamma}{\rho(1-\alpha)} \log(h).$$

The HJB equation in terms of $f$ can be derived from

$$\begin{aligned} V_k(k,h) &= f'(x)xk^{-1}, \\ V_h(k,h) &= \tfrac{1-\alpha+\gamma}{1-\alpha}\big(\rho^{-1}h^{-1} - f'(x)xh^{-1}\big). \end{aligned}$$

After simplifying and collecting terms we finally obtain

$$\rho f(x) + 1 - \tfrac{B(1-\alpha+\gamma)}{\rho(1-\alpha)} + \log f'(x) = \tfrac{B(1-\alpha+\gamma)}{1-\alpha} x \left( \frac{\varphi^{\frac{1-\alpha}{\alpha}} f'(x)^{\frac{1}{\alpha}}}{(\frac{1}{\rho} - f'(x)x)^{\frac{1-\alpha}{\alpha}}} - f'(x) \right), \tag{9}$$

where the constant $\varphi$ is given by

$$\varphi := \Big(\frac{A(1-\alpha)^{2-\alpha}\alpha^\alpha}{B(1-\alpha+\gamma)}\Big)^{\frac{1}{1-\alpha}} > 0.$$

We claim that a solution to this equation is given by

$$f(x) := \frac{B^{\frac{1-\alpha+\gamma}{1-\alpha}} + \rho\log(\rho) - \rho}{\rho^2} + \frac{1}{\rho}\log(x+\varphi). \tag{10}$$

Indeed, we have $f'(x) = 1/(\rho x + \rho\varphi)$ and thus

$$\rho f(x) + 1 - B^{\frac{1-\alpha+\gamma}{\rho(1-\alpha)}} + \log f'(x) = 0,$$

as well as $\rho^{-1} - f'(x)x = \varphi f'(x)$ and hence

$$\frac{\varphi^{\frac{1-\alpha}{\alpha}} f'(x)^{\frac{1}{\alpha}}}{(\frac{1}{\rho} - f'(x)x)^{\frac{1-\alpha}{\alpha}}} = f'(x).$$

We infer that a candidate for the value function is given by:

$$W(k,h) = \frac{B^{\frac{1-\alpha+\gamma}{1-\alpha}} + \rho\log(\rho) - \rho}{\rho^2} + \frac{1}{\rho}\log\big(k + \varphi h^{\frac{1-\alpha+\gamma}{1-\alpha}}\big), \quad k,\, h > 0. \tag{11}$$

However, note that $\lim_{k\to 0} W(k,h) = W(0,h) > -\infty = V(0,h)$ holds, which is contradictory to $\tau = 0$. In any case, this function $W$ is an upper bound for the true value function $V$:

$$V(k,h) \leq W(k,h), \quad \forall\, k,h > 0.$$

The appendix presents a proof of this fact.

Moreover, if for some $(k_0, h_0)$ the pair $(c_t^*, u_t^*)$, derived from the first order conditions, is in $\mathcal{X}$ and $\tau = \infty$ holds, then all inequalities in the proof become equalities, this pair is the optimal control and $V(k_0,h_0) = W(k_0,h_0)$ holds. We insert the controls derived from $W$

$$c^* = \rho(x+\varphi)\frac{k}{x}, \quad u^* = \left(\frac{A(1-\alpha)^2}{B\varphi(1-\alpha+\gamma)}\right)^{\frac{1}{\alpha}} x = \left(\frac{B(1-\alpha+\gamma)}{A\alpha(1-\alpha)}\right)^{\frac{1}{1-\alpha}} x \tag{12}$$

into the dynamics equation (8) for $x_t$:

$$\dot{x}_t = \left(\frac{B(1-\alpha+\gamma)}{(A\alpha)^{\frac{1}{2-\alpha}}(1-\alpha)}\right)^{\frac{2-\alpha}{1-\alpha}} x_t^2 + \left(\frac{B(1-\alpha+\gamma)}{\alpha} - \rho\right)x_t - \rho\left(\frac{A(1-\alpha)^{2-\alpha}\alpha^\alpha}{B(1-\alpha+\gamma)}\right)^{\frac{1}{1-\alpha}} \tag{13}$$

A search for steady states of $x_t$ shows that on the positive axis $\dot{x}_t$ only vanishes for the value

$$x^{ss} := \rho\left(\frac{A(1-\alpha)^{2-\alpha}\alpha}{B^{2-\alpha}(1-\alpha+\gamma)^{2-\alpha}}\right)^{1/(1-\alpha)} = \frac{\rho\alpha\varphi}{B(1-\alpha+\gamma)}.$$

This steady state $x^{ss}$ leads to the balanced growth path, for which the controls $q^{ss} = c_t^{ss} x^{ss}/k_t$ and $u^{ss}$ derived from $W$ remain constant and are thus admissible as long as $u^* \leq 1$ holds.

Linearizing the right hand-side of equation (13) at $x = x^{ss}$ shows that $x^{ss}$ is locally unstable and we infer that $W$ yields the unstable solution branch.

**Proposition 1.** *If* $x^{ss} := \frac{k(0)}{h(0)} = \frac{\rho\alpha\varphi}{B(1-\alpha+\gamma)}$ *and* $\rho\frac{1-\alpha}{1-\alpha+\gamma} \leq B$ *hold, then the controls*

$$c_t^{ss} = \rho\frac{x^{ss}+\varphi}{x^{ss}}k_t = \rho\big(k(0)+\varphi h(0)^{\frac{1-\alpha+\gamma}{1-\alpha}}\big)\exp\big((B\tfrac{1-\alpha+\gamma}{1-\alpha}-\rho)t\big), \tag{14}$$

$$u_t^{ss} = u^{ss} = \frac{\rho(1-\alpha)}{B(1-\alpha+\gamma)} \tag{15}$$

*are indeed optimal in* $\mathcal{X}$*. In addition to the control* $u$*, the control-like variable* $q$ *remains constant as well:*

$$q_t^{ss} = q^{ss} = \rho(x^{ss}+\varphi) = \rho\varphi\frac{\rho\alpha + B(1-\alpha+\gamma)}{B(1-\alpha+\gamma)}.$$

The brief proof of this proposition is given in the appendix. The fixed point derived above is the unique balanced growth path equilibrium of the centralized economy. For $\gamma = 0$ we recover the findings of Benhabib and Perli (1994). However since the social planner's Hamiltonian may well be non-concave they focus on the decentralized case. We have shown that the allocation of human capital is split over both production sectors and remains constant on this path. Furthermore, it is possible to show that

$$\frac{c^{ss}}{y^{ss}} = \frac{(1-\alpha)\left(B\left(1-\alpha+\gamma\right)+\alpha\rho\right)}{B\left(1-\alpha+\gamma\right)} \tag{16}$$

holds, i.e. the fraction of output used for consumption is also constant. Under our hypotheses this fraction is strictly smaller than one. For increasing values of $\gamma$, capturing the degree of the external effect, the steady-state consumption quote decreases. Furthermore, we learn that the steady-state allocation of human capital between the two production sectors is negatively related to the degree of the external effect of human capital in goods production captured by the parameter $\gamma$.

## 3.2 Determining the centralized solution

The main problem in solving the reduced HJB equation (7) stems from the fact that it is not explicit in $f'$. For this implicit differential equation standard techniques (Bronstein and Semendjajew, 1987) are used to establish an explicit differential equation for $f'$. Since we can always add suitable constants to $f$ solving (9), we restrict our attention to the homogeneous form $f(x) = G(x, f'(x))$ of (9) where the function $G$ is given by

$$G(x,p) := -\rho^{-1}\log(p) + \frac{x}{u^{ss}}\big(\psi p^{\frac{1}{\alpha}}(\rho^{-1}-px)^{\frac{\alpha-1}{\alpha}} - p\big) \tag{17}$$

with $\psi := \varphi^{(1-\alpha)/\alpha}$. The function $G$ equals up to an additive constant and the factor $\rho$ the Hamiltonian of the transformed DOP. We find for the derivatives

$$G_x(x,p) = \tfrac{1}{u^{ss}}p\Big(\psi(\rho^{-1}p^{-1}-x)^{\frac{-1}{\alpha}}\big(\rho^{-1}p^{-1}+\tfrac{1-2\alpha}{\alpha}x\big)-1\Big), \tag{18}$$

$$G_p(x,p) = -\rho^{-1}p^{-1} + \frac{\psi x}{u^{ss}}(\rho^{-1}p^{-1}-x)^{\frac{-1}{\alpha}}\big(\alpha^{-1}\rho^{-1}p^{-1}-x\big) - \frac{x}{u^{ss}} \tag{19}$$

By the relationship of $G$ with the Hamiltonian, the Pontryagin maximum principle states for $p_t := f'(x_t)$

$$\dot{x}_t = -p_t + \frac{\psi x_t\big(\alpha^{-1}p_t^{-1} - \rho x_t\big)}{u^{ss}(\rho^{-1}p_t^{-1} - x_t)^{\frac{1}{\alpha}}} - \frac{\rho x_t}{u^{ss}}, \tag{20}$$

which can also be easily verified from equation (8) directly .

Due to $f = G(x, f')$ we get $f' = G_x(x, f') + G_p(x, f')f''$. Thus, setting $p(x) := f'(x)$, we arrive at the explicit differential equation in $p$

$$p'(x) = \frac{p(x) - G_x(x, p(x))}{G_p(x, p(x))},$$

which in our case yields

$$p' = p\frac{u^{ss} + 1 - \psi(\rho^{-1}p^{-1} - x)^{\frac{-1}{\alpha}}(\rho^{-1}p^{-1} + \frac{1-2\alpha}{\alpha}x)}{-x - u^{ss}\rho^{-1}p^{-1} + \psi x(\rho^{-1}p^{-1} - x)^{\frac{-1}{\alpha}}(\alpha^{-1}\rho^{-1}p^{-1} - x)}.$$

The optimal consumption rate satisfies $c^* = V_k^{-1} = k/(xf')$, such that considering $q(x) = cx/k = f'(x)^{-1} = p(x)^{-1}$, the rescaled consumption rate, which we have already encountered in (8), we obtain a differential equation for this control-like variable in terms of the state-like variable $x$:

$$q' = \frac{-p'}{p^2} = q\frac{u^{ss} + 1 - \psi(\rho^{-1}q - x)^{\frac{-1}{\alpha}}(\rho^{-1}q + \frac{1-2\alpha}{\alpha}x)}{x + u^{ss}\rho^{-1}q - \psi x(\rho^{-1}q - x)^{\frac{-1}{\alpha}}(\alpha^{-1}\rho^{-1}q - x)}. \tag{21}$$

This equation is now explicit in $q'$ and standard analytical and numerical methods can be used for its study. Since we know the value of $q$ at the steady-state initial condition $x = x^{ss}$, we face a classical initial value problem where the solution is usually unique. Here, however, uniqueness fails because the candidate function $W$ as well as the true value function both solve the initial value problem. This is due to the fact that at $(x^*, q(x^*))$ in the fraction appearing in equation (21) both numerator and denominator vanish and the right-hand side is indeterminate.

We proceed as follows. The differential equation can be written as

$$q'(x) = \frac{K(x, q(x))}{L(x, q(x))} \quad \text{with} \quad K(x^{ss}, q(x^{ss})) = L(x^{ss}, q(x^{ss})) = 0.$$

In order to obtain determinacy at $x^{ss}$ we use L'Hôpital's rule, which gives

$$q'(x^{ss}) = \frac{K_x(x^{ss}, q(x^{ss})) + K_q(x^{ss}, q(x^{ss}))q'(x^{ss})}{L_x(x^{ss}, q(x^{ss})) + L_q(x^{ss}, q(x^{ss}))q'(x^{ss})}.$$

This leads us to a quadratic equation in $q'(x^{ss})$, one solution of which we already know from $W$, namely $q'(x^{ss}) = \rho$. Therefore, the other solution is given by

$$q'(x^{ss}) = \rho^{-1}\frac{-K_x(x^{ss}, q(x^{ss}))}{L_q(x^{ss}, q(x^{ss}))}.$$

This fraction is now determinate and

$$K_x = \tfrac{-(1-\alpha)\psi}{u^{ss}\alpha^2} q(\rho^{-1}q - x)^{-(1+\alpha)/\alpha}(2\alpha\rho^{-1}q + (1-2\alpha)x),$$

$$L_q = \rho^{-1} + \tfrac{\psi(1-\alpha)}{u^{ss}\alpha^2\rho^2} xq(\rho^{-1}q - x)^{-(1+\alpha)/\alpha},$$

$$\frac{-K_x}{\rho L_q} = \frac{(1-\alpha)\psi q(2\alpha\rho^{-1}q + (1-2\alpha)x)}{u^{ss}\alpha^2(\rho^{-1}q - x)^{(1+\alpha)/\alpha} + \psi(1-\alpha)\rho^{-1}xq}$$

implies

$$q'(x^{ss}) = \rho\Big(1 + \frac{2(u^{ss})^{-1}(1-\alpha)^2 + \alpha(1-\alpha)}{1-\alpha^2 + u^{ss}\alpha}\Big). \tag{22}$$

Note that this value is always larger than the other root $\rho$. This is explained by the fact that this solution, corresponding to the true value function, will run through the origin and thus has to be smaller than the first solution on the interval $[0, x^{ss})$.

In sum, using the differential equation (21) and the steady-state values found in Proposition 1, we can determine the values of the control-like variable $q$ at the state-like values $x$. From this we deduce the corresponding values of $f'$, $c$ and $u$ as well as of $f$ and $V$ for specified initial values $x_0$ or $h_0$ and $k_0$, respectively. Though uniquely determined, the stable solution branch can only be approximated locally by the linearization given in (22) or globally by a numerical solver.

# 4 The decentralized economy

In the decentralized economy the agents have no incentive to exploit the equality of $h$ and $h_a$ stated in equation (4) when deriving their decision rules. The representative agent can write down his dynamic optimization problem (DOP):

$$U = \max_{\{c_t,u_t\}_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-\rho t} \log(c_t)\, dt,$$

with respect to the state dynamics

$$\begin{aligned} \dot{k}_t &= Ak_t^{\alpha}(h_t u_t)^{1-\alpha} h_{a,t}^{\gamma} - c_t, && \forall\, t \geq 0, \\ \dot{h}_t &= B(1-u_t)h_t, && \forall\, t \geq 0, \\ k_t &\geq 0 \quad \text{and} \quad h_t \geq 0 && \forall\, t \geq 0, \end{aligned}$$

given the path of $h_{a,t}$. The initial values $k_0$, $h_0$, $h_{a,0} > 0$ of the state variables are given as well. As in the previous case we can use the homogeneity in the initial conditions of the HJB equation in order to determine an implicit ordinary differential equation. The structure of this equation is the same as before, but it now depends on an exogenous parameter $u_a$.

As in the centralized case, we can transform the problem into solving one explicit ordinary differential equation assuming $u_a$ to be given. This is done in the second paragraph of this section. For given values of $x$ and $u_a$ we can determine the optimal controls $c = c(x, u_a)$ and $u = u(x, u_a)$. Since the model postulates that $u_a$ equals the representative agent's $u$, we vary $u_a$ such that the fixed point $u(x, u_a) = u_a$ holds. It turns out that as long as $\gamma < \alpha$ holds, this equation has always a unique solution. Hence, we have solved the optimization problem also for the decentralized economy and can present some simulation results in the next section.

### 4.1 The representative agent's optimization problem

Our first step is again to define the value function for the DOP at hand. The two controls $c_t$ and $u_t$ are chosen by the representative agent such that they are in $X$ and maximize his discounted utility while taking the economy's average level of human capital $h_{a,t}$ as given. For his optimization he only assumes that $h_{a,t}$ does not grow faster than exponentially in time. The state equations are not time-homogeneous, which is why in the dynamic programming approach the value function is considered for general initial times $t$, not only $t = 0$, and the set of admissible controls is given by

$$(c(\cdot), u(\cdot)) \in \mathcal{X}_t := \{(f, g) : [t, \infty) \to X \,|\, f,\, g \text{ locally bounded and measurable}\}.$$

In order to remain close to the notation in Section 3, the discounting of the value function over time is cancelled by the factor $e^{\rho t}$. Thus, the dynamic optimization problem reads as follows[4]:

$$\tilde{V}(k_t, h_t, t) := \max_{(c,u)\in\mathcal{X}_t} \begin{cases} e^{\rho t} \int_t^\infty \log(c(s)) e^{-\rho s}\, ds, & \tau_t = \infty \\ -\infty, & \tau_t < \infty. \end{cases}$$

The parameter $\tau_t$ denotes the stopping time:

$$\tau_t := \inf\{s \geq t \,|\, k_s = 0\}.$$

The corresponding HJB equation is now also time-dependent:

$$\rho \tilde{V}(k, h, t) = \max_{(c,u)\in X} \big( \log(c) + \tilde{V}_k(k, h, t)\dot{k}_t + \tilde{V}_h(k, h, t)\dot{h}_t + \tilde{V}_t(k, h, t) \big). \quad (23)$$

As before, we can show that a solution $\tilde{W}(k, h, t)$ of this equation, is always an upper bound for the true value function $\tilde{V}(k, h, t)$, as long as $h_{a,t}$ increases at most exponentially and $\tilde{W}$ is concave in $k$ and $h$. Again the proof can be found in the appendix. We proceed as for the social planner's problem. The first order conditions are similar:

$$c^* = \tilde{V}_k^{-1}, \quad (24)$$

$$u^* = \left( \frac{A(1-\alpha)\tilde{V}_k}{B\tilde{V}_h} \right)^{\frac{1}{\alpha}} \frac{k h_{a,t}^{\frac{\gamma}{\alpha}}}{h}. \quad (25)$$

[4] The tilde stresses that we consider the value function of a representative agent and not the central planner's value function.

The interpretation of these first order necessary conditions is the same as in the centralized case and we therefore proceed with the insertion of these findings into the time-dependent HJB equation (23). We obtain

$$\rho\tilde{V}+1=-\log(\tilde{V}_k)+\alpha k\left(A\tilde{V}_k h_{a,t}^{\gamma}\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{B\tilde{V}_h}\right)^{\frac{1-\alpha}{\alpha}}+B\tilde{V}_h h+\tilde{V}_t. \tag{26}$$

Introducing the state-like variable $x_t := k_t h_t^{-1} h_{a,t}^{-\gamma/(1-\alpha)}$ and the control-like variable $q_t := c_t h_t^{-1} h_{a,t}^{-\gamma/(1-\alpha)}$ gives as before

$$\dot{x}_t = A x_t^{\alpha} u_t^{1-\alpha} - q_t - B(1-u_t)x_t - \tfrac{\gamma \dot{h}_{a,t}}{(1-\alpha)h_{a,t}} x_t. \tag{27}$$

We therefore write

$$\tilde{V}(k,h,t)=\tilde{f}\big(kh^{-1}h_{a,t}^{\frac{-\gamma}{1-\alpha}}\big)+\tfrac{1}{\rho}\log\big(hh_{a,t}^{\frac{\gamma}{1-\alpha}}\big).$$

The derivatives of $\tilde{V}$, expressed in terms of the redefined function $\tilde{f}$ and the state-like variable $x$, are

$$\begin{aligned}
\tilde{V}_k(k,h,t) &= \tilde{f}'(x)xk^{-1}\\
\tilde{V}_h(k,h,t) &= -\tilde{f}'(x)xh^{-1}+\rho^{-1}h^{-1}\\
\tilde{V}_t(k,h,t) &= -\tfrac{\gamma}{1-\alpha}\tilde{f}'(x)xh_{a,t}^{-1}\dot{h}_{a,t}+\tfrac{\gamma}{\rho(1-\alpha)}h_{a,t}^{-1}\dot{h}_{a,t}.
\end{aligned}$$

Hence, we arrive at:

$$\begin{aligned}
\rho\tilde{f}(x)+1 = -\log\tilde{f}'(x) &+ \frac{A^{\frac{1}{\alpha}}\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\tilde{f}'(x)^{\frac{1}{\alpha}}x}{B^{\frac{1-\alpha}{\alpha}}(\rho^{-1}-\tilde{f}'(x)x)^{\frac{1-\alpha}{\alpha}}}\\
&+B\big(\rho^{-1}-\tilde{f}'(x)x\big)+\tfrac{\gamma}{1-\alpha}(\rho^{-1}-\tilde{f}'(x)x)\frac{\dot{h}_{a,t}}{h_{a,t}}.
\end{aligned}$$

Consequently, the HJB equation is again reduced. Here, we consider a family of ordinary differential equations depending on the values $h_{a,t}^{-1}\dot{h}_{a,t}$. At this point, after the optimization, we use

$$\dot{h}_{a,t} = B(1-u_{a,t})h_{a,t}, \quad t\geq 0, \qquad u_{a,t}=u_t^*, \tag{28}$$

which is implied by symmetry ($h_{a,t}=h_t$). Substituting the expression (25) for $u_t^*$ would yield wrong results because this would introduce a dependence of $h_{a,t}^{-1}\dot{h}_{a,t}$ on $x$ and $f$ in the differential equation. We must still treat this term as exogenous. We obtain

$$\begin{aligned}
\rho\tilde{f}(x)+1-\tfrac{B(1-\alpha+\gamma-\gamma u_{a,t})}{\rho(1-\alpha)}+\log\tilde{f}'(x)&\\
=\tfrac{B(1-\alpha+\gamma-\gamma u_{a,t})}{1-\alpha}x&\left(\frac{\varphi_a^{\frac{1-\alpha}{\alpha}}}{(\frac{1}{\rho}-\tilde{f}'(x)x)^{\frac{1-\alpha}{\alpha}}}-\tilde{f}'(x)\right),
\end{aligned} \tag{29}$$

where $\varphi_a$ depends on the value $u_a = u_{a,t}$:

$$\varphi_a := \Big(\frac{A(1-\alpha)\alpha^\alpha}{B(1-\alpha+\gamma-\gamma u_a)^\alpha}\Big)^{\frac{1}{1-\alpha}} > 0.$$

Note that $\varphi_a$ depends continuously and monotonously on $u_a$, the values of which lie in $[0,1]$. A slightly modified version of function (10) namely

$$\tilde{f}(x) = \frac{B\frac{1-\alpha+\gamma-\gamma u_a}{\rho(1-\alpha)} + \rho\log(\rho) - \rho}{\rho^2} + \frac{1}{\rho}\log(x+\varphi_a) \tag{30}$$

is a solution of this reduced HJB equation, which only yields admissible controls for the steady state $x^{ss}$ of equation (27), which is given by

$$x^{ss} = \frac{\rho\alpha\varphi_a}{B(1-\alpha+\gamma-\gamma u_a)}. \tag{31}$$

The corresponding optimal controls are found to be

$$u^{ss} = \frac{\rho}{B} \text{ and } q^{ss} = \rho(x^{ss}+\varphi_a) = \rho\varphi_a\frac{\rho(\alpha-\gamma)+B(1-\alpha+\gamma)}{B(1-\alpha+\gamma)-\rho\gamma}. \tag{32}$$

Compared to the centralized case, the steady state allocation of human capital between the two sectors is not influenced by the external effect of human capital in goods production captured by $\gamma$. This tells us that along the balanced growth path the agent has no incentive to take the external effect into account: human capital just grows at a constant rate and thus affects the productivity in the goods sector in a constant manner. Hence, for the allocation of human capital along the balanced growth path, the degree of the external effect of human capital in goods production plays no role in the decentralized economy. On a transition path to the steady state however, the value of $\gamma$ is important since the growth rate of human capital changes over time. Therefore, the value of $\gamma$ is important in order to measure the effect on the productivity in goods production.

Note that the same values for $x$ in the centralized and the decentralized solution yield different capital growth rates because on the balanced growth path human capital will grow with the rate $B-\rho\frac{1-\alpha}{1-\alpha+\gamma}$ and the slower rate $B-\rho$, respectively. The evolution of the decentralized economy is therefore rather to be compared to an economy without external effect, i.e. $\gamma = 0$.

## 4.2 Determining the decentralized solution

In order to determine the optimal control $q(x, u_a)$ given a certain level of $u_a$ we set $\psi$ equal to $\psi_a = \varphi_a^{(1-\alpha)/\alpha}$ in the explicit differential equation (21). The derivation for this equation remains valid using this definition. On the other hand, for the derivative of $q$ at $x^{ss}$ instead of the simplified expression (22) the fraction for $-K_x/(\rho L_q)$ found two lines earlier must be used. Hence, for given $u_a$ the optimal controls can be found by the same methods as before, that is by linear approximation at the steady state or by global numerical approximation. It remains to take care of the condition $u_a = u$. Note that the only possible steady state value of $u_a$ is $\rho/B$ since $u$ has this steady state value independent of $u_a$. That $u_a = u = \rho/B$ holds along the balanced growth path meets the results of Benhabib and Perli (1994), which were derived by different methods.

Off the balanced growth path the representative agent has to find optimal controls meeting the first order conditions (24) and (25). The implicit differential equation (29) is restricted to a particular realization of the parameter $u_a$. For a given level of $x$ every $u_a$ implies certain values of $\tilde{f}'$, $c$ and $u$ as well as of $\tilde{f}$ and $\tilde{V}$. Hence, there is a continuum of possibilities indexed by $u_a$ and a fixed point problem has to be solved.

Given the parameter $u_a$ and the state $x$ the agent chooses his human capital allocation $u$. At this point the symmetry stated in equations (4) and (28) respectively comes into play. The symmetry implies that in the next moment the economy's average allocation of human capital $u_a$ must now equal $u$. Hence the agent has to rethink his decision and so forth. Mathematically speaking, $u$ and $u_a$ will converge with infinite speed (i.e. jump) to a stable fixed point.

Let us consider two different values $u_a^1 < u_a^2$ of $u_a$. Given a certain state we are in, how will the respective optimal controls $u^1$ and $u^2$ be related? We know that the external effect $h_a$ in the second case grows more slowly, hence investing in human capital will not pay as much as in the first case and $u^2 > u^1$ will be chosen. This explains why $u(x, u_a)$ is increasing in $u_a$.

This is corroborated in the left part of Figure 1. The function $u(u_a)$ and the identity are plotted for the degree of the external effect $\gamma = 0.1$ where for the state $x = 1$ holds. The intercept gives us the value $u_a(1) = 0.7522$ for which $u(x, u_a) = u_a$ holds at $x = 1$. In the right part of Figure 1 the function $u_a(x)$ is shown. For a fine grid of points $x$ the intercept has been determined using interval bisection on the domain $[0, 1]$. Note that Brouwer's fixed point theorem (e.g., Rudin (1991)) guarantees the existence of a solution of the equation $u(u_a, x) = u_a$ for each $x$ since the differential equation for $f$ and consequently also those for $q$ and $u$ depend continuously on $\varphi_a$, hence on $u_a$.

Benhabib and Perli (1994) analyze the model with the more general version of the isoelastic utility function where the parameter $\sigma$ stands for the inverse of the intertemporal elasticity of substitution. Setting $\sigma$ equal to one gives the model studied here and their local results show for the decentralized economy that there still is a unique transitional path as long as we are in the neighborhood of the steady state. This result is due to their local approach. For small values of $\gamma$, i.e. not much larger than $\alpha$, our simulation experiments confirm this finding also off the balanced growth path. However for large values of $\gamma$ there may be more than one solution with $u_a = u$. Our global approach shows that multiple transition paths can occur in this model.

Figure 2 shows that in the case of a very strong impact of the external effect, i.e. large $\gamma$, the fixed point is not unique anymore. For fixed value of $x$ the optimal control $u$ then has a large curvature as a function of the average value $u_a$, which yields in fact three fixed points with the largest being equal to one. Among these fixed points the smallest and the largest are stable in the sense

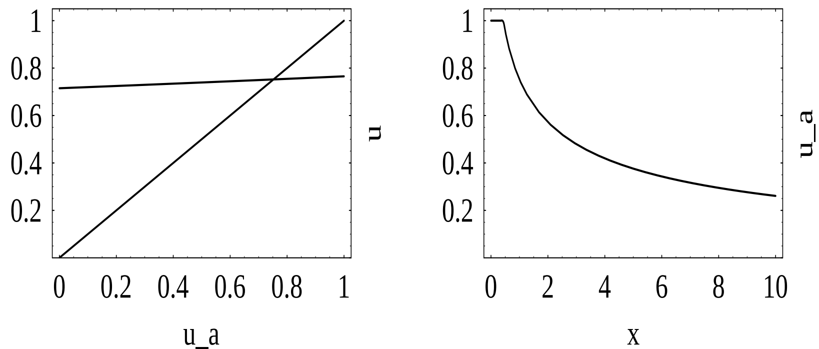


Figure 1: left: optimal time share $u$ with respect to $u_a$ at $x = 1$
right: the fixed point value of $u_a = u$ depending on $x$

that, given $u_a$ in a neighborhood of this fixed point, the control $u$ to be chosen is strictly closer to the fixed point so that $u$ and $u_a$ will be immediately drawn back to the fixed point. While the smallest fixed point is the one that converges to the balanced growth path equilibrium value $u = u_a = \rho/B$, the fixed point $u = u_a = 1$ yields interesting transition dynamics. Here, the agents do not invest in their human capital, but only in physical capital. As a consequence the physical to human capital ratio $x$ increases rapidly. At some point, only the smallest fixed point remains ($u(u_a, x)$ is decreasing in $x$) and the optimal value $u$ jumps immediately to this fixed point value. From there the transition dynamics evolve as usually. It might even happen that we start with a ratio $x_0 < x_{ss}$, keep $u = 1$ fixed until some $x_t > x_{ss}$ is reached before finally converging to $x_{ss}$ from above. Interestingly, this means that given $x = x_{ss}$ and $u_a = 1$ it is possible to choose $u = 1$ and thereby pushing $x$ above its steady state value. At some point the agents start setting $u < 1$, hence forcing $u_a$ to decline. Finally, the decentralized economy will converge to its unique steady state equilibrium with $x = x_{ss}$ and $u = u_a = \rho/B$ established in equation (32).

An economic explanation of this phenomenon is as follows: Suppose that $u_a = 1$ holds, i.e. $\dot{h}_{a,t} = B(1-u_{a,t})h_{a,t} = 0$. The agents behave as if total factor productivity remains constant over some time before it starts rising rapidly. Due to the strong external effect ($\gamma$ large) and the relative lack of physical capital ($x$ small) there is a high incentive for all agents to invest in physical capital as much as possible ($u = 1$) in order to benefit from future growth of total factor productivity. The physical capital investment lowers the marginal productivity of human capital in goods production and after some time investment in human capital is performed: $u < 1$ is chosen. At this time very rapidly the agents observe that the average human capital increases and thus it pays even more to defer production of physical capital and $u$ is reduced further. Very quickly a new equilibrium (fixed point) is reached.

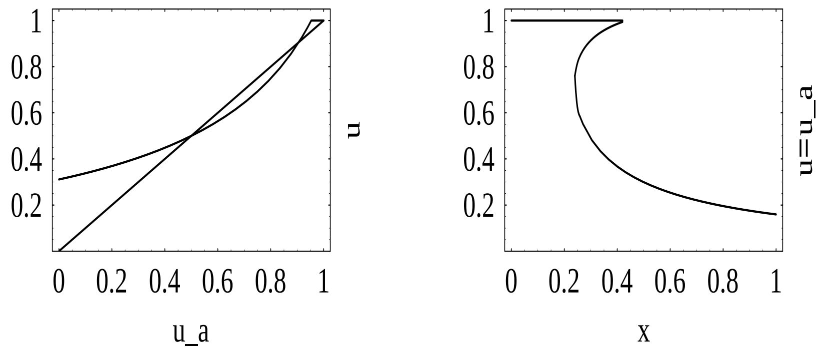


Figure 2: left: *multiple equilibria* - $u$ with respect to $u_a$ at $\gamma = 5$
right: *bifurcation* - the fixed point $u = u_a$ with respect to $x$

# 5 Simulation results

In the previous sections we have reduced the problem of determining the value function to solving an explicit ordinary differential equation in $q$ with value $q(x^{ss})$ prescribed by the balanced growth path solution. Although an analytical expression of this solution is not known, an easy and fast Euler scheme can be used to determine the function $q$ numerically. This is a clear advantage compared to the approach via the Pontryagin maximum principle adopted by Benhabib and Perli (1994) as well as Bond, Wang, and Yip (1996), which yields – even after reduction – three nonlinear coupled dynamic equations where only the values for $t \to \infty$ are known from the balanced growth path. The more demanding techniques of backward solving for this kind of problems as advocated by Brunner and Strulik (2002) are thus avoided.

In this section we make use of our findings and show that on the transition path the output growth rate towards the balanced growth rate obeys a hump-shaped course. Based on an annual approach we consider the following typical calibration of the parameter values:

$$A = 1, \quad B = \tfrac{1}{10}, \quad \rho = \tfrac{1}{20}, \quad \alpha = \tfrac{1}{3}. \tag{33}$$

Figure 3 shows the phase diagram of the centralized economy for $q(x)$ setting $\gamma = 0.1$. The straight line is the function $q(x) = \rho(x + \varphi)$ which is derived from the solution $W$, whereas the concave function starting in the origin is the optimal control $q$ derived from the true value function $V$, that is the numerical solution of the differential equation (21). Both functions meet in the saddle point $(x^{ss}, q^{ss})$. The concave dotted line indicates the curve where the function $L = L(x, q)$ vanishes. By the relationship (20) this corresponds to the values of $(x, q)$ where $\dot{x}_t = 0$ holds. Above this line the derivative $\dot{x}_t$ is negative and below it is positive. Geometrically speaking, in the upper region $x_t$ moves to the left and in the lower region it moves to the right. The flatter dotted line corresponds to the values of $(x, q)$ where $\dot{q}_t = 0$ holds. Above this line $q_t$ shrinks and below it increases. This shows again that the linear control $q$ derived from $W$ corresponds to the unstable solution branch, whereas the numerically determined optimal control $q$ is indeed globally stable and induces an adjustment to the balanced growth path solution. Qualitatively, the phase diagram shows the same features as for other classical growth models, cf. Chapter 16 in Intrilligator (1971). Since the main implications of the centralized and the decentralized economy are qualitatively similar we concentrate on the first with $\gamma = 0.1$ from now on.

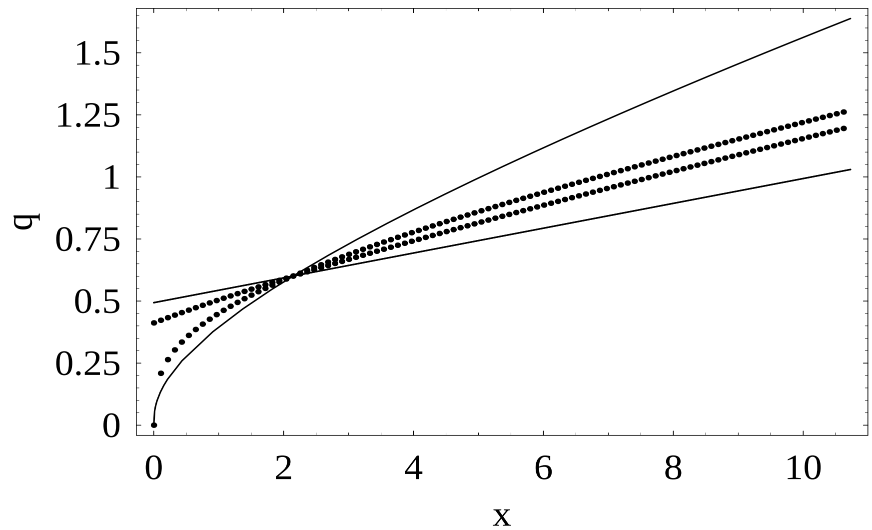


Figure 3: Phase diagram for $(q, x)$ with saddle point

Figure 4 displays the growth rate of output with respect to the state-like variable $x$ and time $t$ respectively. Analytically this rate can be subdivided as follows:

$$\frac{\dot{y}_t}{y_t} = \alpha \frac{\dot{x}_t}{x_t} + (1-\alpha)\frac{\dot{u}_t}{u_t} + \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_t}{h_t}. \tag{34}$$

Let us discuss the terms in the sum separately as functions of $x$. The first term $\dot{x}_t/x_t$ is positive for $x < x_{ss}$ and negative for $x > x_{ss}$, tending to infinity for $x \to 0$, see Figure 5 in the Appendix. From the decay of $u(x)$ we infer that the second term $\dot{u}_t/u_t$ has the reverse property: it is negative for $x < x_{ss}$ and positive for $x > x_{ss}$. The last term $\dot{h}_t/h_t$ is never negative, monotonically increasing with $\lim_{x\to\infty} \dot{h}/h = B$ and $\dot{h}(x_{ss})/h(x_{ss}) = B - \rho$. Hence, the three terms behave very differently and yield in sum the surprising hump shape of $\dot{y}/y_t$ that we observe.

The left of Figure 4 shows the simulation results for the growth rate of output with respect to the state variable $x$. The horizontal line denotes the steady state growth rate. The right intersection of both lines corresponds to the balanced growth path.

Large values of $x$ correspond to the case when human capital is relatively scarce. The agent invests in schooling activities in order to generate more human capital. The resulting high growth rate of human capital explains the fact that the output growth rate is above the balanced growth rate. At the same time the agent may consume more goods than are currently produced, implying disinvestment in physical capital and thus a following of the output growth rate. Then human capital growth converges to the steady state and the output growth rate imitates this behavior.

Small values of $x$ correspond to a scarcity in physical capital. The agent can not simply "eat" human capital in order to be on the balanced growth path. Although he may wish to set $u$ bigger than one, i.e. he would like to increase productivity in the goods sector even if it costs him negative human capital growth, he must wait until the physical capital stock achieves the corresponding size. Let us assume that $u$ is currently equal to one. Since the agent would like to accumulate physical capital we know that the output stream is bigger than the consumption stream. However, if $u$ is equal to one, the goods technology has diminishing returns in physical capital and therefore the additional unit of $k$ lowers marginal productivity in goods production. The output growth rate declines and falls even below the balanced growth rate. When the restriction $u \leq 1$ is not binding anymore the agent starts to shift human capital to the schooling sector and the decline of the marginal productivity of physical capital is slowed down. This effect is the stronger the more human capital is shifted to the schooling sector. At some point, due to the growing stock of human capital the marginal productivity of physical capital even starts to rise again. The output growth rate just imitates this behavior and finally converges to the steady state growth rate. The hump-shaped course is even more pronounced when we calibrate the productivity parameter $B$ smaller.

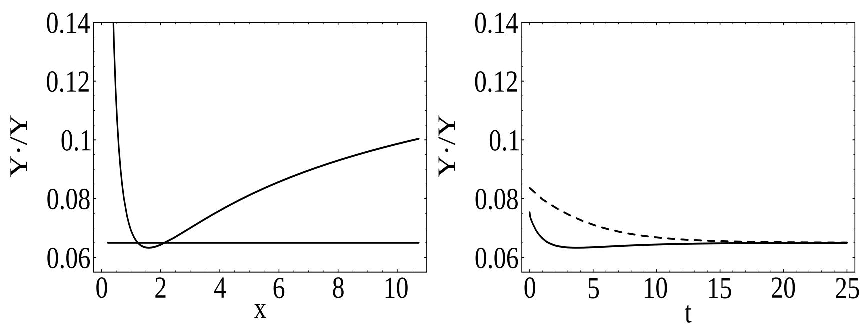


Figure 4: left: Growth rate of output with respect to the state x
right: Growth rate of output with respect to time (years)

The right of Figure 4 shows the simulation results for the growth rate of output with respect to time. The dashed curve corresponds to the case when human capital is relatively scarce. The agent invests in schooling activities in order to generate more human capital. The high growth rates of human capital together with its high marginal productivity in goods production explain the fact that the output growth rate is above the balanced growth rate thereby falling over time. The solid line starts with a scarcity in physical capital and the same reasoning as above explains the reversed hump-shaped course. High growth rates of physical capital together with its high but decreasing marginal productivity explain the high but falling output growth rate. The effect is that $k$ grows slower and slower. After some time $h$ starts to grow. This latter effect counteracts the lowering of the marginal productivity of $k$. After some time both effects exactly cancel out. Thereafter the second effect even dominates the first one and the growth rate slowly converges from below to its steady state value.

The above result is in sharp contrast to the neoclassical growth model, where it is well known that the output growth rate falls the higher the physical capital stock $k$ is. Consider equation (34) without the $\dot{u}/u$ and $\dot{h}/h$ terms. Interpreting $x$ as the physical capital stock then refers to the neoclassical growth model and the growth rate is positive if $x$ is smaller than its steady state value $x_{ss}$, negative if $x$ is bigger than $x_{ss}$, and zero if $x = x_{ss}$ holds, i.e. the reversed hump-shaped course does not show up in the neoclassical growth model.

# 6 Conclusion

In this paper, we have introduced an easy way of analyzing the transitional dynamics of the Uzawa-Lucas endogenous growth model with logarithmic utility. We have used the value function approach to solve both the social planner's optimization problem and the representative agent's optimization problem in the decentralized economy. By exploiting the homogeneity of the problem we are able to reduce the complexity of the Hamilton-Jacobi-Bellman equation significantly. We therefore merely have to solve a one-dimensional differential equation in the parameters of the model. Our method applies in both cases, the centralized and the decentralized economy. The only difference between the two cases is that the representative agent has to deal with a family of ordinary differential equations indexed by $u_a$. Therefore, he has to solve a fixed point problem such that there is no contradiction between the representative agent's and the average choice of $u$ or $u_a$, respectively. Since our method works in the neighborhood of the steady state as well as on the transition path, it has global character.

The Uzawa-Lucas endogenous growth model allows for a positive external effect of human capital such that the second welfare theorem may not always hold. The steady state relationships (15) and (32) show that in the presence of an external effect, i.e. $\gamma$ is strictly positive, the planner's allocation of human capital to the schooling sector is bigger than the representative agent would have chosen in a decentralized economy. As our simulations show, this is also true off steady state. The failure of the second welfare theorem in this model gives a possible reasoning for indeterminacy that has been studied for example by Benhabib and Perli (1994) or Xie (1994). For the planner's solution, we show numerically that there is a unique transitional path to the steady state no matter where we start.

The global character of our method enables us to find multiple transition paths to the unique steady state so far undetected in the literature even if $x = x_{ss}$ holds. We find parameter constellations where there are two possibilities with $u = u_a$ solving the Hamilton-Jacobi-Bellman equation (29). As Xie (1994) argues this phenomenon may explain why we observe the process of "lagging behind, catching up with, and overtaking". This insight unveils the shortcoming of local treatments prevailing in economic theory. Furthermore our simulations show that the growth rate of output obeys a hump-shaped course, which contradicts the predictions of the neoclassical model. This finding can provide a theoretical explanation for non-monotonous rates of convergence between different countries observed empirically.

# Appendix

# A Proofs

## A.1 $W$ is an upper bound for the true value function $V$

For all $k_0, h_0, t > 0$ and all controls $(c_t, u_t) \in \mathcal{X}$ with $\tau > t$ we have

$$\begin{aligned} e^{-\rho t} W(k_t, h_t) &= W(k_0, h_0) + \int_0^t (-\rho e^{-\rho s} W(k_s, h_s) \\ &\qquad + e^{-\rho s} W_k(k_s, h_s) \dot{k}_s + e^{-\rho s} W_h(k_s, h_s) \dot{h}_s)\, ds \\ &\leq W(k_0, h_0) + \int_0^t (-\rho e^{-\rho s} W(k_s, h_s) \\ &\qquad + e^{-\rho s} (\rho W(k_s, h_s) - \log(c_s))\, ds \\ &= W(k_0, h_0) + \int_0^t e^{-\rho s} (-\log(c_s))\, ds, \end{aligned}$$

where we have used that $W$ solves the HJB equation or is at least an upper bound, i.e. a supersolution, if $(c_t^*, u_t^*) \notin \mathcal{X}$ since we have not excluded the values $u_t^* > 1$. Hence, by rearranging the terms and taking the limit $t \to \infty$, we obtain

$$V(k_0, h_0) \leq W(k_0, h_0) - \lim_{t \to \infty} e^{-\rho t} W(k_t, h_t),$$

if the latter limit exists. Note that this inequality is always trivially valid for $\tau < \infty$. Since $h_t$ grows exponentially and $c_t \geq 0$ holds, also $k_t$ cannot grow faster than exponentially and thus $W(k_t, h_t)$ grows at most linearly, which implies that the limit cannot be larger than zero. On the other hand, $h_t \geq h_0$ holds for all $t \geq 0$ so that $W(k_t, h_t)$ is uniformly bounded from below for all $t \geq 0$. Hence, this last limit exists, equals zero, that is the transversality condition is fulfilled, and

$$V(k_0, h_0) \leq W(k_0, h_0)$$

holds, as asserted.

### A.2 Proposition 1

Obviously, the controls $c^{ss}$ and $u^{ss}$ are admissible because of $u^{ss} \leq 1$ by assumption. The value of $x^{ss}$ ensures that we are on the balanced growth path and $\dot{x}_t = 0$ holds so that the values of the controls are easily derived from (12). Hence by the preceding remark on $W$, the controls are indeed optimal.

### A.3 $\tilde{W}$ is an upper bound for the true value function $\tilde{V}$

For all $k_v, h_v, t > v$ and any controls $(c_s, u_s) \in \mathcal{X}_v$ with $\tau_v = \infty$ we find

$$\begin{aligned} e^{-\rho t} W(k_t, h_t, t) &= e^{-\rho v} W(k_v, h_v, v) + \int_v^t e^{-\rho s} \big( -\rho W(k_s, h_s) + W_k(k_s, h_s, s) \dot{k}_s \\ &\quad + W_h(k_s, h_s, s) \dot{h}_s + W_t(k_s, h_s, s) \big) \, ds \\ &\leq e^{-\rho v} W(k_v, h_v, v) + \int_v^t (-\rho e^{-\rho s} \log(c_s)) \, ds. \end{aligned}$$

The exponential growth bounds for $h_{a,t}$ and $h_t$ imply exponential bounds for $k_t$ and $c_t$ so that $\lim_{t \to \infty} e^{-\rho t} W(k_t, h_t, t) = 0$ is guaranteed. We infer

$$W(k_v, h_v, v) \geq e^{\rho v} \int_v^\infty e^{-\rho s} \log(c_s) \, ds.$$

Since the controls were arbitrary, we have shown $W(k, h, v) \geq V(k, h, v)$ under the only hypothesis $\tau_v = \infty$. If $\tau_v < \infty$ were true, the value function would equal $-\infty$ and the asserted inequality is trivially true.

## B More simulation results

### B.1 The state-like variable $x$

In the centralized economy the state-like variable $x_t$ adjusts to the balanced growth path with exponential speed, but the rate gets slower the closer to the steady state values we are, see Figure 5 left hand side. Again, the dashed line corresponds to a scarcity in human capital while the solid refers to a scarcity in physical capital. Since the agent is not allowed to disinvest in human capital but very well in physical capital, the model obeys an asymmetry. This asymmetry causes that the adjustment of $x_t$ proceeds faster for $x_t > x_{ss}$. In the decentralized economy the knowledge of $u_a$ allows us to simulate the adjustment of the $x_t$ over time. The same phenomenon as for the central planner's solution occurs.

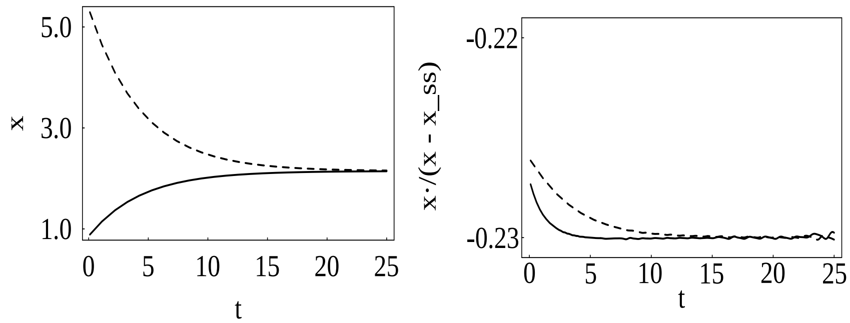


Figure 5: Absolute and relative adjustment of $x_t$ over time

Next, we consider the relative adjustment rate $\frac{\dot{x}_t}{x_t - x^{ss}}$. We find, that the closer $x_t$ is to its limiting value $x^{ss}$, the faster the relative adjustment is. For large values of $t$ small oscillations can be seen, which are artefacts due to the numerical approximation. In this region $x_t - x^{ss}$ is already of order $10^{-10}$ and the numerical precision fades. Neglecting this effect, the rate is seen to converge for $t \to \infty$ against the model's stable eigenvalue. This is typical for growth models and confirms our calculations. Let us now turn to the control-like variables.

## B.2 The optimal choice of $u$

Figure 6 shows the value of $u$ in the $(x, \gamma)$ space as a surface. The black line corresponds to the steady state values $(x^{ss}, u^{ss})$. Keeping $\gamma$ fixed, the fraction of time allocated to the goods production decreases when $x$ increases. Similarly $u$ decreases when the external effect $\gamma$ of human capital in goods production increases, given a certain level of $x$. The first observation can be explained as follows. A high value of $x$ indicates that the economy's endowment with human capital is relatively low. This circumstance causes a high marginal productivity of human capital in the goods sector. Arbitrage reasoning implies that the realized marginal productivity of human capital in the schooling sector determined by $B\,(1 - u_t)$ must also be relatively high. Hence a comparable high fraction of human capital is attracted by the schooling sector. This explains the relatively low value of $u_t$. The second effect occurs because in the presence of a higher external effect the central planner finds it more attractive to accumulate human capital in order to exploit the higher social return of human capital in goods production. The second reasoning does also hold for the influence of $\gamma$ on the steady state value $u^{ss}$ described in Section 3.1.

For small values of $x$ and $\gamma$ the planner is about to set $u$ outside the region where $u \in [0, 1]$ holds. There we have to set $u = 1$ and solve the optimization problem for $c$, keeping $u = 1$ fixed. Mathematically, this corresponds to a free boundary problem. At this point we want to mention that we have studied the difference in the solution shown above and a hypothetical solution where the planner is allowed to chose $u$ freely from $[0, \infty)$, i.e. the planner may temporarily wish to disinvest in human capital. We found that the values of the control-like variable $q$ for both solutions are very similar. The planner arranges the consumption pattern in such a way that the disadvantage of the binding constraint $u \leq 1$ is borne by all points in time. Therefore the hypothetical consumption path is smoothed and very close to the true path. This follows directly from the constant elasticity of substitution of degree one implied by the logarithmic utility function.

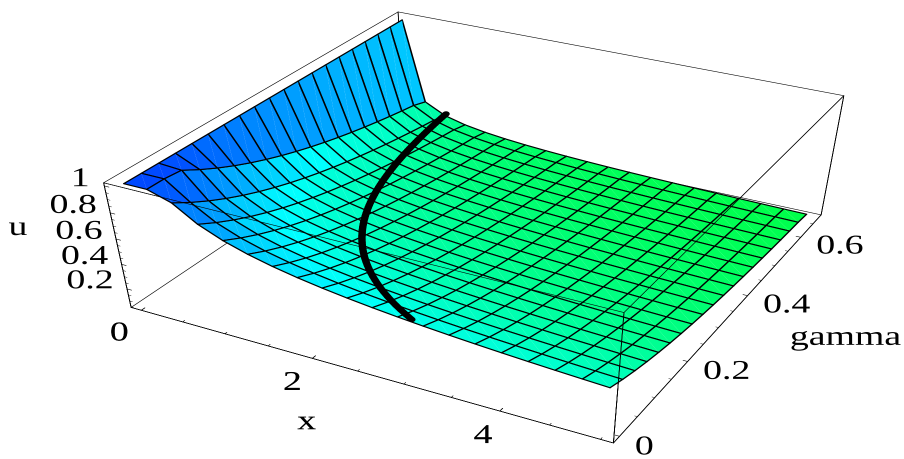


Figure 6: optimal time share $u$ with respect to $(x, \gamma)$

## B.3 The control-like variable $q$

Figure 7 shows the value of the control-like variable $q$ in the $(x, \gamma)$ space. Holding $\gamma$ constant, we see that $q$ increases with $x$. Since we have a single good economy, this effect is very intuitive. For fixed $\gamma$ we can draw a line from the point $(q, x) = (0, 0)$ and any point on the surface for the same $\gamma$. By concavity, this line lies below the surface, hence telling us that although $c$ and $k$ move in the same direction, $k$ moves faster. If we consider a point on the surface where $x_t \leq x^{ss}$ holds, the slope of the line which is equal to the economy's $\frac{c}{k}$ ratio is higher than in the steady state. The consumption level increases slowly, while the physical capital stock increases slightly faster. However, the difference in both rates becomes more and more negligible as the economy reaches the steady state. An equivalent reasoning is true when $x_t \geq x^{ss}$ holds. Here, the two variables are decreasing, the consumption level in a slower manner than the stock of physical capital. Further simulation results show us that for very high values of $x$ the consumption level may even be higher than the output flow. These are the regions where $u$ is very small and the planner tries to accumulate as much human capital as possible even if the current consumption stream is generated by disinvestment in physical capital. This again shows the way the constant elasticity of intertemporal substitution drives the consumption decisions. Even a temporary disinvestment of physical capital is taken into account in order to smooth the consumption path.

The dark line in Figure 7 is again the projection of the steady state value $q^{ss} = q(x^{ss})$ onto the surface. It shows that the steady state ratio of consumption and physical capital increases with $\gamma$. This is due to the curvature of the surface. We find that the model implies realistic consumption capital ratios. Empirically it is well known that the consumption quote is about 60 percent[5], while equation (16) implies values between $\frac{5}{6}$ for $\gamma = 0$ and $\frac{7}{9}$ for $\gamma = \frac{1}{3}$ and is never below $\frac{2}{3}$ for $\gamma \to \infty$. The numbers for $\frac{k}{y}$ are hard to estimate and therefore not very exact. We found values between 1.8 for the United Kingdom and 3.0 for Japan[6]. Together these numbers imply that $\frac{c}{k}$ should lie around $\frac{1}{4}$. As can be seen in Figure 7, our simulation replicates this ratio for the steady state values of $\frac{c_t}{k_t}$, i.e. for $\frac{q^{ss}}{x^{ss}}$.

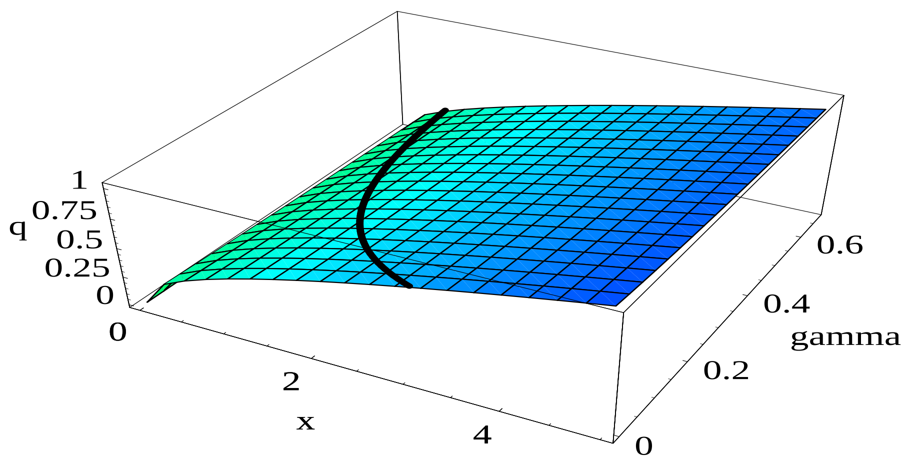


Figure 7: optimal consumption $q$ with respect to $(x, \gamma)$

[5]See for example Burda and Wyplosz (2001). Using IMF data they find values between 56.2 percent for Germany and 64.9 percent for the USA.

[6]See Maddison (1995) cited in Burda and Wyplosz (2001).

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